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THE GROWTH AND DECAY OF A FROZEN LAYER IN FORCED FLOW

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Abstract—The problem of the unsteady behavior of a frozen layer in a liquid flowing past a cold thick (plane) wall is studied analytically. The integral (profile) method is utilized, which takes full account of the movement of the phase conversion front and transient heat conduction within the wall. The growth of the frozen layer comes to a stop when the conduction heat flux into the wall balances convection from the liquid. At this instant, the frozen layer will begin to melt; it ultimately disappears when melting is complete. Numerical predictions of the maximum frozen layer thickness, the time the frozen layer passes through the maximum, and the total lifetime of the layer are obtained displaying the principal effects of two governing dimensionless parameters.

NOMENCLATURE

- c, specific heat;
- h, convective heat-transfer coefficient;
- k, thermal conductivity;
- L, latent heat of fusion;
- q, heat flux to crust surface;
- t, time;
- T, temperature;
- T_0, initial temperature of wall;
- y, distance measured from wall-crust interface.

Greek symbols

- α, thermal diffusivity;
- β, dimensionless heat of fusion, equation (12);
- δ, crust thickness;
- Δ, dimensionless crust thickness, equation (9);
- δ_T, thermal wave thickness;
- Δ_T, dimensionless thermal wave thickness, equation (10);
- δ_s, crust thickness on isothermal wall at steady state, equation (17);
- θ_i, dimensionless wall–crust interface temperature, equation (8);
- λ, crust growth constant, equation (20);
- ρ, density;
- σ, $k\rho c$ ratio, equation (13);
- τ, dimensionless time, equation (11);
- τ_{life}, dimensionless crust lifetime;
- ψ, temperature profile shape factor, equation (6).

Subscripts

- i, at wall–crust interface;
- max, at maximum crust thickness;
- mp, equilibrium melting point;
- ∞, in flowing liquid far from crust surface;
- 1, frozen crust material;
- 2, wall material.

1. INTRODUCTION

THIS paper is concerned with the transient behavior of the frozen layer (or "crust") that forms when a warm liquid suddenly flows over the face of a semi-infinite solid body that is at a temperature below the freezing point of the liquid. This type of solidification problem arises, for example, in the freezing of a lava stream flowing over rock or soil. The application that motivated the present study is the freezing of molten steel or UO_2 fuel in the cold, thick-walled channel regions surrounding a liquid–metal fast reactor core (following a hypothetical core-disruptive accident). The prediction of ablation of the channel wall or flow blockages by excessive solidification requires an understanding of the transient behavior of the frozen layer on the channel wall. The present investigation is limited to the flow along a cold plane wall of infinite extent in the direction normal to the flow. The solidification process is initiated by suddenly allowing the warm liquid to flow over the wall. Normally the average temperature of the liquid will be higher than its solidification temperature. As the frozen crust on the wall increases in thickness, heat is transferred by convection from the flowing liquid to the solid crust–liquid interface. Heat is removed from the solidified layer by transient conduction through the cold wall. Solidification proceeds as long as the conduction heat flux through the frozen crust to the wall exceeds that to the crust surface from the flowing liquid. Once the thermal boundary layer in the wall is sufficiently large, the conduction heat flux into the cold wall exactly balances convection from the liquid and there is no latent heat generated. The crust reaches a maximum size. After this, however, convective heat transfer becomes more than what can be conducted away, resulting actually in reduction in thickness through melting. The rate of melting increases with time because the conduction heat flux into the cold wall becomes increasingly smaller. Ultimately, melting will cause the crust to disappear. The crust thickness–time history $\delta(t)$ constitutes the result sought. In particular, we are concerned here with the following three questions: What is the maximum crust thickness? At what value of time, t, does the crust thickness peak? At what value of time does the crust disappear via melting? Physically, these features represent the potential for flow blockage or wall ablation in channel flow.

There is an extensive literature on the solidification of flowing liquids on a plane wall [1–8], with most predictions being confined to cases of (i) an isothermal cold wall or (ii) a thin wall of negligible heat capacity which is cooled by a coolant liquid flowing along the other side of the wall. In [8], the heat capacity of the wall is accounted for, but the wall is sufficiently thin so that the growing solid layer thermally "communicates" with the opposite side of the wall early in the transient. In these solidification problems, the frozen layer never melts but, instead, approaches a steady-state thickness. The results of the calculations for an isothermal wall appear to be a special case of the solutions for the thick (plane) wall of finite thermal conductivity presented here.

A continuous metal casting process known as "dip-forming" does involve freezing followed by melting of the frozen layer [9, 10].* In this process a cool metal bar is passed continuously through a bath of superheated molten metal. Some metal freezes onto the bar as it enters the bath. If the height of the bath is great, the frozen metal crust will begin to melt. In [11] an analysis is given of the dip-forming process. Under most conditions of interest the bar is sufficiently thin so that the crust growth and decay behavior depends mostly on the heat it takes to raise the temperature within the bar uniformly from initial temperature to melting temperature.

2. PHYSICAL MODEL

A liquid at a fixed bulk temperature T_∞ above its solidification temperature T_{mp} suddenly flows over a thick cold wall at an initially uniform temperature $T_0 (T_0 < T_{mp})$† As shown in Fig. 1, a solidification front propagates into the liquid from $y = 0$ at a rate determined both by the convective heat flux, q, from the liquid to the phase boundary and the rate at which the heat of solidification can be conducted from the front. Specifically, we make the following simplifying assumptions:

A1. The solidified layer is thin compared with its extension in the direction of flow so that heat conduction in this direction is small compared to that normal to the flow.

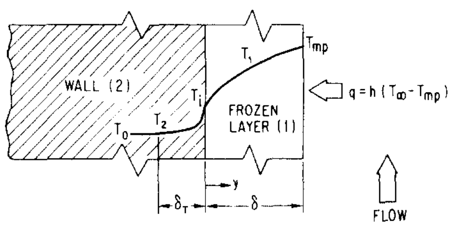

FIG. 1. Schematic of frozen layer on semi-infinite wall, indicating instantaneous temperature profile and nomenclature.

A2. All physical properties (density, heat capacity, thermal conductivity) are considered constant. In addition, densities of liquid and solid deposit are taken to be the same.

A3. The Solidification (or melting) front is sharp and planar on the scale of the crust thickness, and thermal equilibrium exists at the phase conversion front.

A4. The convective heat-transfer coefficient between the flowing liquid and phase boundary remains constant with time and, therefore, the liquid supplies a constant convective heat flux $h(T_\infty - T_{mp})$ to the crust surface at all times. While h is considered constant for each element of frozen layer, it may vary with distance along the cold surface.

A5. Simple analytic forms (polynomials) can be assumed for the temperature profiles in the solidified layer and wall regions, provided they are consistent with the boundary conditions and the overall heat balance requirements. A thermal boundary layer of thickness δ_T is assumed to propagate into the cold wall.

A6. In many cases occurring in practice, the thermal boundary layer will proceed only a relatively small distance into the wall; consequently, the cold wall may be considered to extend to infinity in the direction normal to the flow (i.e. in the negative y-direction).

The validity of assumptions A1–A4 was experimentally demonstrated for the case of solidification of warm water flowing over a thin plate cooled from below in [12]. Assumption A4, in which thermal convection in the liquid is taken to be known and unchanged due to thickness variation of the solid phase, has been invoked in previous analytical studies involving solidification with forced convection [1–8, 11] and melting with free convection [13]. Assumption A5 allows the governing partial differential equations and boundary conditions to be replaced by an approximate first order system of ordinary differential equations using the well-known integral (moment) method [14] (see Section 3).

3. ANALYSIS

Subject to the above assumptions, conditions at the moving phase boundary ($y = \delta$), at the solidified layer-wall interface ($y = 0$), and across the solidified layer (subscript 1) and wall region (subscript 2) suffice to provide the desired relationship between $\delta(t)$ and the parameters of the problem. These conditions are:

$$T_1(0, t) = T_2(0, t) = T_i(t) \text{ (temperature continuity)} \quad (1)$$

*The author is indebted to an anonymous reviewer for calling attention to the literature on dip-forming.

†It is assumed here that steady-state flow conditions are achieved in the liquid before any measurable crust growth occurs. A transient solidification process could be initiated by considering an instantaneously cooled wall submerged in an otherwise isothermal steady flow of liquid. However, in many physically realistic situations, it is difficult to imagine freezing to begin in this manner, especially on a very thick wall.

$$k_1\left(\frac{\partial T_1}{\partial y}\right)_{y=0} = k_2\left(\frac{\partial T_2}{\partial y}\right)_{y=0} \quad \text{(heat flux continuity)} \quad (2)$$

an energy balance which equates the instantaneous latent heat generation $\rho_1 L\,\mathrm{d}\delta/\mathrm{d}t$ to the conductive heat loss to the crust minus the convective heat transport to the phase boundary, i.e.

$$\rho_1 L\frac{\mathrm{d}\delta}{\mathrm{d}t} = k_1\left(\frac{\partial T_1}{\partial y}\right)_{y=\delta} - h(T_\infty - T_{\mathrm{mp}}) \quad (3)$$

a macroscopic heat balance across the solidified layer

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\delta T_1(y,t)\,\mathrm{d}y = \alpha_1\left(\frac{\partial T_1}{\partial y}\right)_{y=\delta} - \alpha_1\left(\frac{\partial T_1}{\partial y}\right)_{y=0} + T_{\mathrm{mp}}\frac{\mathrm{d}\delta}{\mathrm{d}t} \quad (4)$$

and a macroscopic heat balance across the thermal boundary layer in the cold wall

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\delta_T}^0 T_2(y,t)\,\mathrm{d}y = \alpha_2\left(\frac{\partial T_2}{\partial y}\right)_{y=0} - \alpha_2\left(\frac{\partial T_2}{\partial y}\right)_{y=-\delta_T} + T_0\frac{\mathrm{d}\delta_T}{\mathrm{d}t}. \quad (5)$$

Equations (1)–(5) constitute five conditions which the temperature profiles must satisfy. If we assume second-order polynomials for the temperature distributions $T_1(y,t)$ and $T_2(y,t)$, they take the forms

$$\frac{T_1(y,t) - T_{\mathrm{mp}}}{T_i(t) - T_{\mathrm{mp}}} = \psi(t)\left[1 - \frac{y}{\delta(t)}\right] + [1-\psi(t)]\left[1 - \frac{y}{\delta(t)}\right]^2 \quad (0 < y \leqslant \delta) \quad (6)$$

where $\psi(t)$ is a shape function eliminated in the course of the analysis (in favor of the remaining dependent variables), and

$$\frac{T_2(y,t) - T_0}{T_i(t) - T_0} = \left[1 + \frac{y}{\delta_T(t)}\right]^2 \quad (-\delta_T < y < 0). \quad (7)$$

It should be noted that equation (7) is consistent with the notion of a thermal wave thickness, beyond which the effects of the phase change process are negligible. Moreover, in writing equations (6), (7) we have already invoked the condition of temperature continuity at $y = 0$.

It is now convenient to introduce the following set of dimensionless variables and parameters:

$$\theta_i \equiv \frac{T_i - T_0}{T_{\mathrm{mp}} - T_0}$$

(dimensionless crust–wall interface temperature) (8)

$$\Delta \equiv \frac{h(T_\infty - T_{\mathrm{mp}})}{k_1(T_{\mathrm{mp}} - T_0)}\delta$$

(dimensionless crust thickness) (9)

$$\Delta_T \equiv \left(\frac{\alpha_1}{\alpha_2}\right)^{1/2}\frac{h(T_\infty - T_{\mathrm{mp}})}{k_1(T_{\mathrm{mp}} - T_0)}\delta_T$$

(dimensionless thermal wave thickness) (10)

$$\tau \equiv \frac{h^2(T_\infty - T_{\mathrm{mp}})^2}{k_1^2(T_{\mathrm{mp}} - T_0)^2}\alpha_1 t$$

(dimensionless time, Fourier number) (11)

$$\beta \equiv \frac{L}{c_1(T_{\mathrm{mp}} - T_0)}$$

(dimensionless latent heat, phase change number) (12)

$$\sigma \equiv \left(\frac{k_2\rho_2 c_2}{k_1\rho_1 c_1}\right)^{1/2} \quad (k\rho c \text{ ratio}). \quad (13)$$

Then, from equations (2)–(7), the following system of dimensionless equations is obtained.

$$\frac{\mathrm{d}\Delta_T}{\mathrm{d}\tau} = \frac{6}{\Delta_T} - \frac{\Delta_T}{\theta_i}\frac{\mathrm{d}\theta_i}{\mathrm{d}\tau} \quad (14)$$

$$\beta\frac{\mathrm{d}\Delta}{\mathrm{d}\tau} = \frac{2(1-\theta_i)}{\Delta} - \frac{2\sigma\theta_i}{\Delta_T} - 1 \quad (15)$$

$$\tau + (1+\beta)\Delta - \tfrac{1}{3}\Delta(1+2\theta_i) - \frac{\sigma}{3}\theta_i\Delta_T\left[1 + \left(\frac{\Delta}{\Delta_T}\right)^2\right] = 0. \quad (16)$$

Equations (14)–(16) re-express the mathematical formulation of the model and are sufficient to determine the three coupled unknown functions $\Delta_T(\tau;\sigma,\beta)$, $\Delta(\tau;\sigma,\beta)$, and $\theta_i(\tau;\sigma,\beta)$ subject to the initial conditions $\Delta_T = 0$, $\Delta = 0$ when $\tau = 0$. It can be shown that Δ_T, Δ, and θ_i can each be represented by a power series of the form

$$\sum_{n=1}^{\infty} b_n\tau^{n/2};$$

however, this series can only be used for short times. As τ increases the series ceases to converge and becomes oscillating divergent. Therefore, for longer times, including the period in which the crust thickness passes through a maximum, we require a computer solution. For this purpose and to utilize available computer subroutines, equations (14)–(16) were converted to an equivalent coupled system of first-order differential equations by differentiating equation (16). The resulting system was integrated in the forward direction using the power series solutions to obtain the correct starting behavior. Numerical integration was performed using a library program (based on the Gear method [15]) available at the Applied Mathematics Division of Argonne National Laboratory. A typical case ran in about 6 s on the Laboratory Computer Center IBM 370 system.

4. RESULTS AND DISCUSSION

A typical set of crust thickness–time curves is shown in Fig. 2 for several values of the $k\rho c$ ratio (for the case $\beta = 1$). One notes that $\Delta(\tau)$ passes through a maximum

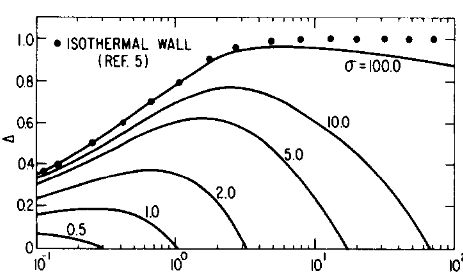

FIG. 2. Effect of $k\rho c$ ratio on crust thickness–time histories ($\beta = 1.0$).

value Δ_{max} at a time τ_{max}. At this extremum, the heat flux coming from the liquid flow is equal to that removed through the wall by conduction. The rate of solidification passes through a stage of zero speed and melting begins. In the case of an isothermal wall ($\sigma \to \infty$), the frozen layer never melts; rather it approaches a steady-state value δ_s which can be obtained by a simple heat balance [5]:

$$\delta_s = \frac{k_1(T_{mp} - T_0)}{h(T_r - T_{mp})}. \tag{17}$$

This thickness has been used here as a reference length so that $\Delta = \delta/\delta_s$ [see equation (9)]. Savino and Siegel [5] have obtained an analytical expression by an iteration technique for the growth of a solid layer on an isothermal wall. This expression is plotted in Fig. 2. As expected, when the $k\rho c$ ratio is very large, the growth behavior initially is that of the growth of a solidified layer on an isothermal wall. Ultimately, however, Δ drops below this asymptote, owing to the effects of heat conduction in the wall. It should be noted that there is good agreement between the integral method and the analytical expression provided by Savino and Siegel [5] in this special case $\sigma \to \infty$. For a given value of β, the $\sigma \to \infty$ case probably constitutes the most stringent test of accuracy of an approximate method since in this case the temperature profile exhibits the greatest curvature. Further discussion on the accuracy of the integral method will be postponed until Section 5.

In Fig. 3 are shown the temperature distributions within the frozen layer and the wall. The dimensionless temperature is shown as a function of position

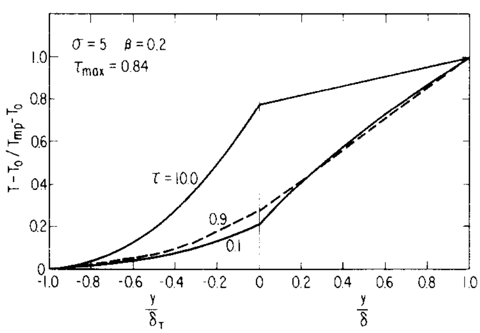

FIG. 3. Effect of time on temperature profile in frozen layer and cold wall. (Profile at $\tau \approx \tau_{max}$ dashed for clarity only.)

normalized by crust thickness for $y > 0$ and thermal boundary-layer thickness for $y < 0$. The dimensionless time is treated as a parameter. Since $\beta = 0.2$ is rather small, we expect a large heat capacity effect. It is interesting to note the behavior of the curves as τ becomes large. Initially the temperature profile within the growing crust exhibits some curvature, due to the effects of heat capacity (energy storage); however, the profile becomes linear as the crust reaches maximum thickness and tends to remain fairly close to linear throughout the melting period. This merely implies that once solidification ends and melting begins, the rate of movement of the phase boundary is very much slower than the rate of propagation of a thermal wave

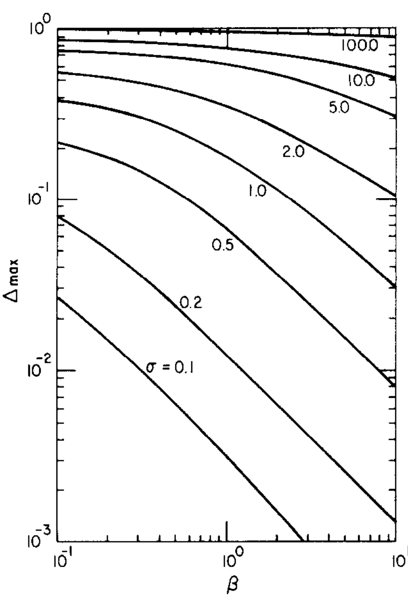

FIG. 4. Maximum crust thickness as a function of heat of fusion parameter β; dependence on $k\rho c$ ratio, σ.

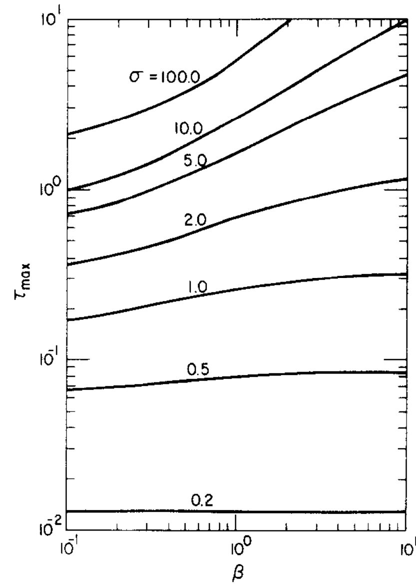

FIG. 5. Time to reach maximum crust thickness as a function of heat of fusion parameter, β; dependence on $k\rho c$ ratio, σ.

within the crust. The temperature distribution within the crust is approximately that corresponding to steady state.

We now focus attention on the $\Delta_{max}(\sigma, \beta)$ and $\tau_{max}(\sigma, \beta)$ relations shown in Figs. 4 and 5. Increasing the $k\rho c$ ratio, σ, is seen to have the expected effects of increasing the value of Δ_{max} and lengthening the solidification period τ_{max} of the $\Delta(\tau)$ trace. Of course, Δ_{max} approaches unity (i.e. $\delta \to \delta_s$) for large values of σ. The predicted behavior of the crust lifetime, $\tau_{life}(\sigma, \beta)$, is shown in Fig. 6. It is noteworthy that τ_{life} is insensitive to the dimensionless latent heat β. This is because during the melting period ($\tau > \tau_{max}$), well

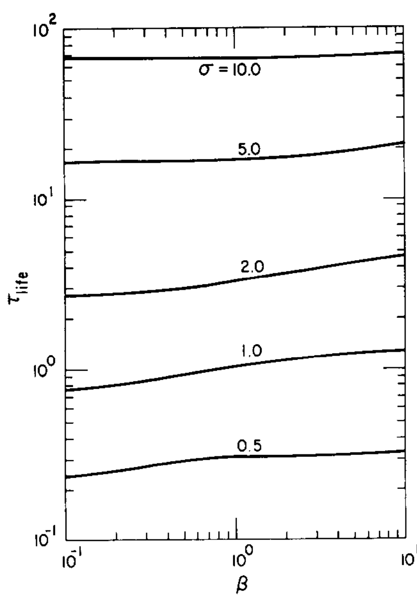

FIG. 6. Crust lifetime as a function of heat of fusion parameter, β; dependence on $k\rho c$ ratio, σ.

before the solid deposit disappears, the initial effects of heat capacity (or temperature profile distortion) within the solid deposit are "forgotten". The convective heat flux from the liquid flow is transmitted directly to the solid deposit-wall interface. The interface temperature T_i at $y = 0$, therefore, approaches the simple asymptotic behavior

$$T_i \sim T_0 + \frac{2h(T_\infty - T_{mp})}{k_2}\left(\frac{\alpha_2 t}{\pi}\right)^{1/2} \tag{18}$$

which is the surface temperature corresponding to constant heat flux at the surface of a semi-infinite solid [16]. Since the solid deposit is completely melted when $T_i \to T_{mp}$, we obtain the following simple estimate for τ_{life}

$$\tau_{life} \sim \frac{\pi}{4}\sigma^2. \tag{19}$$

It should be mentioned that when the warm fluid flows over a thick wall of the same material, the melting front will penetrate the wall when $\tau > \tau_{life}$. The temperature profile within the solid wall will asymptotically approach a simple, steady exponential form [16] involving the melting rate (or wall ablation rate) which, in turn, will become a linear function of the convective heat flux.

Using the results of this section, it is interesting to examine the behavior of an ice layer deposited from a flowing water stream at 20°C onto a steel surface after being chilled in liquid nitrogen ($T_0 = -196$°C).* For this system $\sigma = 6.06$ and $\beta = 0.84$. Figures 4–6 then reveal $\Delta_{max} \approx 6.8$, $\tau_{max} \approx 1.7$, and $\tau_{life} \approx 25.0$. To convert these dimensionless numbers to predicted maximum crust thickness δ_{max}, time t_{max} for δ to peak, and crust lifetime t_{life}, we consider turbulent flow over a flat plate. The convective heat-transfer coefficient in steady turbulent flow is a weak function of the streamwise coordinate. Therefore, the ice growth (and decay) will be very nearly one dimensional. For a free stream water velocity of $10^3\,cm\,s^{-1}$, we obtain $h \approx 0.35$ cal $cm^{-2}\,s^{-1}\,K^{-1}$. This estimate leads to the inferences: $\delta_{max} \approx 1.09$ cm, $t_{max} \approx 4.17$ s, and $t_{life} \approx 61.4$ s.

*An experimental investigation of transient freezing in tube flows has employed water flowing in channels bounded by thick walls initially cooled to the liquid nitrogen boiling temperature [17].

5. ACCURACY OF THE COMPUTATIONAL METHOD AND TRANSIENT CONDUCTION MODEL

5.1. *Accuracy of the integral method*

The integral (profile) method exploited here lends itself well to the treatment of transient conduction problems with moving boundaries [14]. However, in any new application, it is desirable to examine its accuracy by comparison with a closely related exact (or numerical) solution. As usual, these exact solutions pertain to much less general phase change problems than that explicitly treated in Section 4. We have already considered the accuracy of the present method when applied to the growth of a solid deposit in the degenerate case $\sigma \to \infty$ (isothermal wall) in Section 4 for $\beta = 1$. Moreover, the validity of the integral method in this singular case has been demonstrated in [5] and need not be belabored here. Here we examine the growth predictions near $\tau = 0$ for arbitrary σ and β. In addition, the present method is compared to the approximate results of a mathematical model in which the transient term is ignored in solving the conduction equation in the solid deposit.

In the neighborhood of $t = 0$, the interfacial heat transfer due to pure transient conduction overshadows any convection due to liquid flow. In this time domain, the problem reduces to that of a solid layer growing in a stagnant liquid at its solidification temperature. For τ near zero, equations (14)–(16) simplify to yield the growth law

$$\Delta = 2\lambda\tau^{1/2} \tag{20}$$

where λ, the "growth constant", is given by the implicit relation

$$\tfrac{1}{3}\beta\lambda^4 + (\tfrac{4}{3})^{1/2}\frac{\beta}{\sigma}\lambda^3 + (\tfrac{1}{3} + 2\beta)\lambda^2 + (3)^{1/2}\frac{\beta}{\sigma}\lambda - 1 = 0. \tag{21}$$

Equation (21) is obtained by neglecting the last term in equation (15) and assuming solutions of the form $\Delta \sim \tau^{1/2}$, $\Delta_T \sim \tau^{1/2}$, and θ_i = constant. A mathematically exact implicit relationship between the growth constant and the parameters β, σ can be found in [16], viz.:

$$\lambda \exp(\lambda^2)[1/\sigma + \mathrm{erf}(\lambda)] = \frac{1}{\pi^{1/2}\beta}. \tag{22}$$

As shown in Fig. 7, the integral method faithfully predicts the dependence of the growth constant on the $k\rho c$ ratio and dimensionless latent heat. In the parametric extreme $\sigma \to \infty$ and $\beta \ll 1$, the results are especially encouraging, with equation (21) representing the exact growth constant to better than 6.0%.

An approximate solution for the frozen crust behavior can be obtained by neglecting the transient

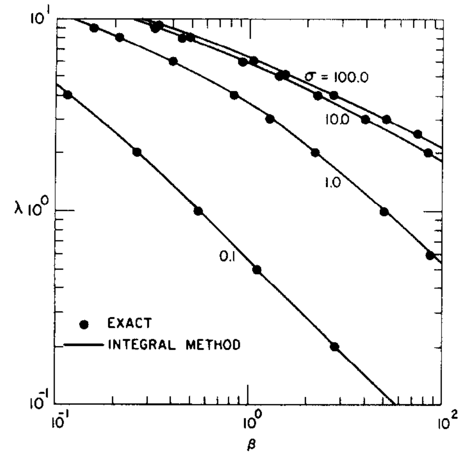

FIG. 7. Accuracy of present integral method when applied to the limiting case τ near zero; comparison of predicted and exact growth constants for various combinations of σ and β.

term in solving the conduction equation in the solid deposit and using the resulting value of $\partial T/\partial y$ at $y = \delta$ in equation (3) to predict the boundary motion. This so-called quasi-steady method, intuitively expected to be valid for "thick" thermal boundary layers in the solid deposit (i.e. for $\beta \gg 1$), leads to the following nonseparable growth–melt law when cast in the notation of Section 3:

$$\beta \frac{\mathrm{d}\Delta}{\mathrm{d}\tau} = \frac{1}{\Delta} \exp\left(\frac{\tau}{\sigma\Delta^2}\right) \operatorname{erfc}\left[\left(\frac{\tau}{\sigma\Delta^2}\right)^{1/2}\right] - 1. \quad (23)$$

But if equation (23) is formally applied when $\beta \gg \sigma$ it can be simplified to the form

$$\beta \frac{\mathrm{d}\Delta}{\mathrm{d}\tau} = \frac{\sigma}{(\pi\tau)^{1/2}} - 1 \quad (\beta \gg 1;\ \sigma \ll \beta). \quad (24)$$

Equation (24) immediately gives the crust thickness–time history:

$$\Delta = \frac{2}{(\pi)^{1/2}} \frac{\sigma}{\beta} \tau^{1/2} - \frac{1}{\beta} \tau \quad (\beta \gg 1;\ \sigma \ll \beta). \quad (25)$$

Comparison of the integral method with this relation (see Fig. 8) reveals good agreement in this special case $\beta \gg 1$ and $\sigma \ll \beta$.

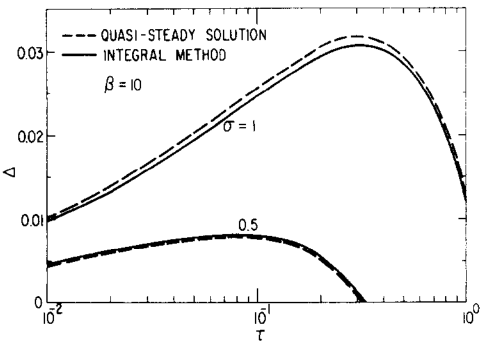

FIG. 8. Comparison of predicted crust thickness–time curves for the limiting case $\beta \gg 1$, $\beta \gg \sigma$.

5.2. *Conduction model*

Most of the assumptions underlying the present model (e.g. constant physical properties, one-dimensional heat conduction, and well-defined phase front) have been discussed by others and need not be considered here. Instead, we focus attention on the validity of the assumed constant convective heat-transfer coefficient (cf. Section 2, A4), for which a convenient quantitative criterion has apparently not previously been given. Indeed, thermal convection in the liquid and the thickness variation of the solid phase are both unsteady and coupled to each other. It is reasonable to suppose, however, that convection in the liquid is relatively undisturbed by the moving phase boundary if the time necessary for a thermal conduction wave to span the liquid boundary-layer dimension, $\delta_b \approx k/h$, is much smaller than the time it takes the solidification front to traverse δ_b. This leads us to expect negligible transient effects in the liquid flow when $\lambda \ll 1$ or when $\beta \gg 1$ (see Fig. 7).* When β is not large, however, it is not obvious that the liquid transient will, in turn, exert much influence on the behavior of the frozen layer. The maximum transient temperature and velocity profile distortions must occur at a very short time after the commencement of the freezing process since the phase boundary velocity is greatest then. It is during this initial period that the flow process has little effect upon the freezing process. A detailed treatment of this liquid flow transient is beyond the scope of this paper. Suffice it to say that it entails the solution of a non-similar transient boundary layer problem with the heat transfer determined simultaneously with the present analysis of the behavior of the frozen layer.

6. CONCLUSIONS

Using the integral method, we have studied the (one-dimensional) behavior of a frozen layer that forms on a cold, thick wall in a flowing warm liquid, taking into account transient heat conduction within the wall. The behavior of the dimensionless frozen layer is shown to depend on two parameters. One parameter is essentially a dimensionless latent heat of fusion. The second is the ratio of the transient conduction heat flow resistance of the wall to that of the frozen layer. The frozen layer grows until it reaches a maximum thickness less than the steady-state thickness obtained for growth on an isothermal wall. After this the frozen layer undergoes a reduction in thickness via melting; it continues to melt until it disappears. Numerical predictions for the maximum thickness, the time the crust thickness peaks, and the total lifetime of the frozen layer are displayed graphically.

In particular systems of experimental interest, it may be necessary to relax the one-dimensional assumption so the treatment can be applied to the study of the behavior of a frozen layer in a thick-walled tube or in

*Low Prandtl number liquids (i.e. $Pr < 1$) require that $\lambda/(Pr)^{1/2} \ll 1$. This imposes a condition on the ratio of the solidification front speed to the momentum wave speed in the liquid boundary layer.

a thick-walled rocket-motor nozzle.* Moreover, in fast reactor safety applications, the surface of the wall may melt upon contact with the warm flowing liquid [19, 20] or during the course of the freezing–melting transient. Both of these extensions are readily handled using the heat balance integral (profile) method.

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*It has been suggested by Ungar [18] that the performance of a nozzle structure during a solid-propellant rocket-motor firing is intimately related to the growth and decay of a frozen oxide layer along the interior nozzle surface.

DEVELOPPEMENT ET FUSION D'UNE COUCHE DE CONGELATION EN ECOULEMENT FORCE

Résumé—On étudie analytiquement le problème du comportement instationnaire d'une couche congélée dans un liquide s'écoulant sur une paroi épaisse (plane) et froide. La méthode intégrale (du profil) est utilisée, elle tient pleinement compte du mouvement du front de changement de phase et de la conduction transitoire de chaleur dans le mur. Le développement de la couche congelée s'arrête lorsque le flux de conduction thermique dans le mur équilibre la convection dans le liquide. A cet instant, la couche congelée va commencer à fondre; elle disparait enfin lorsque la fusion est achevée. On a obtenu des prévisions numériques de l'épaisseur maximale de la couche congelée, de l'instant auquel la couche congelée passe par un maximum, ainsi que de la durée de vie totale de la couche, qui permettent de dégager l'influence de deux paramètres adimensionnels fondamentaux.

DAS ANWACHSEN UND ABSCHMELZEN EINER GEFRORENEN SCHICHT BEI ERZWUNGENER KONVEKTION

Zusammenfassung—Das instationäre Verhalten einer gefrorenen Schicht in einer Flüssigkeit, die entlang einer kalten, dicken ebenen Wand strömt, wird analytisch untersucht. Es wird die integrale Profilmethode herangezogen; sie berücksichtigt die Bewegung der Phasenänderungsfront und der instationären Wärmeleitung innerhalb der Wand. Das Anwachsen der gefrorenen Schicht endet dann, wenn der Wärmefluß aufgrund von Leitung in der Wand aufgewogen wird durch den Konvektionsfluß in die Flüssigkeit. Dann beginnt die gefrorene Schicht zu schmelzen; sie verschwindet letztlich wenn der Schmelzvorgang abgeschlossen ist. Für die größte Dicke der gefrorenen Schicht, für die Zeit, in der die gefrorene Schicht durch ein Maximum geht und für die gesamte Lebensdauer der Schicht werden numerische Beziehungen erhalten, die grundsätzliche Einflüsse von zwei charakteristischen dimensionslosen Parametern zeigen.

РОСТ И РАЗРУШЕНИЕ СЛОЯ ЛЬДА ПРИ ВЫНУЖДЕННОМ ТЕЧЕНИИ

Аннотация — Теоретически исследуется задача нестационарного поведения слоя льда при обтекании жидкостью холодной плоской стенки большой толщины. Используется интегральный метод, который достаточно полно объясняет движение фронта фазового перехода и нестационарную теплопроводность в стенке. Слой льда прекращает рост, когда тепловой поток в стенку уравновешивает конвекцию от жидкости. В этом случае слой льда начинает таять; он полностью исчезает после таяния. Полученные численные расчеты максимальной толщины слоя льда, времени достижения слоем льда максимальной величины и суммарное время жизни слоя отражают основную роль в процессе двух безразмерных параметров.